

# How to Resum Feynman Graphs

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## Abstract

In this paper we reformulate in a simpler way the combinatoric core of constructive quantum field theory. We define universal rational combinatoric weights for pairs made of a graph and one of its spanning trees. These weights are nothing but the percentage of Hepp's sectors in which the tree is leading the ultraviolet analysis. We explain how they allow to reshuffle the divergent series formulated in terms of Feynman graphs into convergent series indexed by the trees that these graphs contain. The Feynman graphs to be used are not the ordinary ones but those of the intermediate field representation, and the result of the reshuffling is called the Loop Vertex Expansion.

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# 1 Introduction

The fundamental step in quantum field theory (QFT) is to compute the logarithm of a functional integral<sup>1</sup>. The main advantage of the perturbative expansion in QFT into a sum of Feynman amplitudes is to perform this computation explicitly: the logarithm of the functional integral is simply the same sum of Feynman amplitudes restricted to *connected* graphs. The main disadvantage is that the perturbative series indexed by Feynman graphs typically diverges. Constructive theory is the right compromise, which allows both to compute logarithms, hence connected quantities, but through convergent series. However it has the reputation to be a difficult technical subject.

Perturbative quantum field theory writes quantities of interest (free energies or connected functions) as sums of amplitudes of connected graphs

$$S = \sum_G A_G. \quad (1)$$

However such a formula (obtained by expanding in a power series the exponential of the interaction and then illegally commuting the power series and the functional integral) is *not* a valid definition since usually, even with cutoffs, even in zero dimension (!) we have

$$\sum_G |A_G| = \infty. \quad (2)$$

This divergence, known since [1], is due to the very large number of graphs of large size. We can say that Feynman graphs *proliferate too fast*. More precisely the power series in the coupling constant  $\lambda$  corresponding to (1) has zero radius of convergence<sup>2</sup>. Nevertheless for the many models built by constructive field theory, the constructive answer is the *Borel sum* of the perturbative series (see [4] and references therein). Hence the perturbative expansion, although divergent, contains all the information of the theory; but it should be *reshuffled* into a convergent process.

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<sup>1</sup>The main feature of QFT is the renormalization group, which is made of a sequence of such fundamental steps, one for each *scale*.

<sup>2</sup>This can be proved easily for  $\phi_d^4$ , the Euclidean Bosonic quantum field theory with quartic interaction in dimension  $d$ , with fixed ultraviolet cutoff, where the series behaves as  $\sum_n (-\lambda)^n K^n n!$ . It is expected to remain true also for the renormalized series without cutoff; this has been proved in the super-renormalizable cases  $d = 2, 3$  [2, 3]).

The central basis for the success of constructive theory is that *trees* do *not* proliferate as fast as graphs<sup>3</sup>, and they are sufficient to see connectivity, hence to compute logarithms. This central fact is not usually emphasized as such in the classical constructive literature [6]. It is also partly obscured by the historic tools which constructive theory borrowed from statistical mechanics, such as lattice cluster and Mayer expansions.

The LVE [7] is a recent constructive technique to reshuffle the perturbative expansion into a convergent expansion using canonical combinatoric tools rather than non-canonical lattices. Initially introduced to analyze *matrix* models with quartic interactions, it has been extended to arbitrary stable interactions [8], shown compatible with direct space decay estimates [9] and with renormalization in simple super-renormalizable cases [10, 11]. It has also recently been used and improved [12] to organize the  $1/N$  expansion [13, 14, 15] for random *tensors* models [16, 17, 18], a promising approach to random geometry and quantum gravity in more than two dimensions [19, 20].

It is natural to ask how Feynman graphs are regrouped and summed by this LVE. The purpose of this paper is to answer explicitly this question. We define a simple but non trivial<sup>4</sup> set of positive weights  $w(G, T)$ , which we call the *constructive* weights. These weights are rational numbers associated to any pair made of a connected graph  $G$  and a spanning tree  $T \subset G$ , which are normalized so that

$$\sum_{T \subset G} w(G, T) = 1. \quad (3)$$

They reduce the essence of constructive theory to the single short equation

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G, T) A_G = \sum_T A_T, \quad A_T = \sum_{G \supset T} w(G, T) A_G. \quad (4)$$

Indeed if we formulate  $S$  in terms of the *right graphs*, then

$$\sum_T |A_T| < +\infty, \quad (5)$$

which means that  $S$  is now well defined!

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<sup>3</sup>This slower proliferation of trees allows for the local existence theorems in classical mechanics, since classical perturbation theory is indexed by trees [5]. Hence understanding constructive theory as a recipe to replace Feynman graphs by trees creates also an interesting bridge between QFT and classical mechanics.

<sup>4</sup>Non-trivial means they are *not* the trivial equally distributed weights  $w(G, T) = 1/\chi(G)$ , where  $\chi(G)$ , the complexity of  $G$ , is the number of its spanning trees.

In the first section of this paper we define the constructive weights  $w(G, T)$  as the percentage of Hepp's sectors [21] in which the tree  $T$  is dominant. We then establish our main new result, Theorem 2.1, namely an integral representation (7) of these weights in terms of the positive-type matrix which is at the heart of the forest formulas of constructive theory [22, 23] and of the LVE [7]. This representation connects Hepp's sectors, the essential tools for renormalization in the parametric representation of Feynman integrals, to the forest formula, the essential tool of the LVE. It strongly suggests that the LVE is well-adapted for renormalization, especially in its parametric representation defined in [12].

In the second section we explain what are the right graphs to use. In the Bosonic case, they are not the ordinary Feynman graphs, but the graphs of the so-called intermediate field representation of the theory. This was the essential discovery of the LVE [7]. In the third section we fully explicit up to second order the corresponding graphs and their reshuffling in the very simple case of the  $\phi_0^4$  quantum field theory in zero dimension. We end up with a conjecture, which, if true, would allow to define QFT in non-integer dimension of space-time.

## 2 The Weights

### 2.1 Paths and Sectors

We consider from now on pairs  $(G, T)$  always made of a *connected* graph  $G$  and one of its *spanning* trees  $T$ . We denote  $V$  the number of vertices and  $E$  the number of edges of  $G$ . Graphs with multiple edges and self-loops (called tadpoles in physics) are definitely allowed, as they occur as Feynman graphs in QFT.

Given such a pair  $(G, T)$  and a pair  $(i, j)$  of vertices in  $G$  there is a unique *path*  $P_{ij}^T$  in  $T$  joining  $i$  to  $j$ . If  $\ell$  is an edge of  $G \supset T$ , we also note  $P_\ell^T$  the unique path in  $T$  joining the two ends  $i$  and  $j$  of  $\ell$ .

A *Hepp sector*  $\sigma$  of a graph  $G$  is an ordering of its edges [21]; hence there are  $E!$  such sectors. For each such sector  $\sigma$  there is a single associated tree  $T(\sigma)$ . This tree is defined recursively as follows. We consider the first edge in the ordering. If it is not a loop edge we keep it in the tree  $T(\sigma)$ , otherwise we don't. After  $k$  steps we have built a forest  $F_k$  of at most  $k$  edges. We consider the next edge  $\sigma(k+1)$  in the ordering  $\sigma$ . We add it to  $F_k$  if it creates

no cycle with the edges of  $F_k$ , otherwise we throw it away. We continue until the last edge has been considered. This process builds a unique spanning tree of  $G$ ,  $T(\sigma)$ , which is the tree in  $G$  with minimal edges with respect to the ordering  $\sigma$ . The tree  $T(\sigma)$  is exactly the leading tree of the Kirchoff-Symanzik polynomial  $U_G$  of the parametric representation ((31)-(32) below) in the Hepp sector  $\sigma$ .

## 2.2 Definitions

There are two equivalent ways to define the constructive weights  $w(G, T)$ , through paths or through sectors. The sector definition is simpler as it simply states that  $w(G, T)$  is the percentage of sectors  $\sigma$  such that  $T(\sigma) = T$ .

**Definition 2.1.**

$$w(G, T) = \frac{N(G, T)}{E!} \quad (6)$$

where  $N(G, T)$  is the number of sectors  $\sigma$  such that  $T(\sigma) = T$ .

From this definition it is obvious that the  $w(G, T)$  form a probability measure for the spanning trees of a graph, hence that (3) holds. It is also obvious that these weights are integers divided by  $E!$ , hence rational numbers. Remark also that the weights  $w(G, T)$  are *symmetric* with respect to relabeling of the vertices of  $T$  (which are also those of  $(G)$ ). However the positivity property important for constructive theory is not obvious in this definition.

**Theorem 2.1.**

$$w(G, T) = \int_0^1 \prod_{\ell \in T} dw_\ell \prod_{\ell \notin T} x_\ell^T(\{w\}) \quad (7)$$

where  $x_\ell^T(\{w\})$  is the infimum over the  $w_{\ell'}$  parameters of the lines  $\ell'$  in  $P_\ell^T$ . If  $\ell$  is a self-loop, hence the path is empty, we put  $x_\ell^T(\{w\}) = 1$ .

**Proof:** We introduce first parameters  $w_\ell$  for all the edges in  $G - T$ , writing

$$x_\ell^T(\{w\}) = \int_0^1 dw_\ell \left[ \prod_{\ell' \in P_\ell^T} \chi(w_\ell \leq w_{\ell'}) \right], \quad (8)$$

where  $\chi(\cdots)$  is the characteristic function of the event  $\cdots$ . Then we decompose the  $w$  integrals according to all possible orderings  $\sigma$ . We need only prove that

$$\begin{aligned} w(G, T) &= \int_0^1 \prod_{\ell \in G} dw_\ell \prod_{\ell \notin T} \left[ \prod_{\ell' \in P_\ell^T} \chi(w_\ell \leq w_{\ell'}) \right] \\ &= \sum_{\sigma} \int_{0 \leq w_{\sigma(E)} \leq \cdots \leq w_{\sigma(1)} \leq 1} \chi(T(G, \sigma) = T). \end{aligned} \quad (9)$$

This is true because in the sector  $\sigma$  the function  $\prod_{\ell \notin T} [\prod_{\ell' \in P_\ell^T} \chi(w_\ell \leq w_{\ell'})]$  is zero or 1 depending whether  $T(G, \sigma) = T$  or not. Hence

$$\int_0^1 \prod_{\ell \in T} dw_\ell \prod_{\ell \notin T} x_\ell^T(\{w\}) = \frac{N(G, T)}{E!}. \quad (10)$$

□

This theorem provides an integral representation of the weights, in terms of "weakening parameters"  $w_\ell$  for the edges  $\ell \in T$ . The fundamental advantage of the constructive weights  $w(G, T)$  over naive uniform weights is precisely the positivity property of the  $x_\ell^T(\{w\})$  matrix, which we now explain.

### 2.3 Positivity

To any triplet  $(G, T, \sigma)$  is associated a sequence of  $V$  partitions  $B_k$ ,  $k = 1, \dots, V$ , of the set of vertices of  $G$  into disjoint blocks, which are the connected components of the sequence of forests obtained when constructing  $T(\sigma)$ . More precisely the first partition  $B_1$  is made of singletons, one for each vertex of  $V$ ; the second partition is made of the connected components of the forest  $F_1$  made of the first edge of  $T(\sigma)$ , and so on until  $B^V$  which is made of a single connected component containing all vertices of  $G$ . Clearly there are exactly  $V - i + 1$  disjoint blocks in  $B_i$ , labeled as  $B_k^a$ ,  $a = 1, \dots, V - i + 1$ . Remark that these partitions only depend on  $\bar{\sigma}$ , the restriction of the ordering  $\sigma$  to  $T$ .

The  $V$  by  $V$  real symmetric block matrix  $B_k(T, \bar{\sigma})_{ij}$  with 1 between elements  $i, j$  belonging to the same connected component  $B_k^a$  and 0 between elements  $i, j$  belonging to different connected component  $B_k^a$  at stage  $k$  is

obviously positive (although not positive definite as soon as blocks are not trivial).

**Theorem 2.2** (Positivity). *Let us consider the  $V$  by  $V$  real symmetric matrix  $x_{ij}^T(\{w\})$  defined like in Theorem 2.1, namely with 1 on the diagonal  $i = j$  and with the infimum over the  $w_{\ell'}$  parameters over the lines  $\ell'$  in  $P_{ij}^T$  for non diagonal entries  $i \neq j$ . This matrix is positive type for any  $w_{\ell} \in [0, 1]^{V-1}$ .*

**Proof:** This is the central property of the forest formulas [22, 23]. We recall briefly the proof here for completeness. Consider a fixed value of the  $w_{\ell} \in [0, 1]^{V-1}$ . There is at least one sector  $\bar{\sigma}$  of  $T$  to which it belongs, hence such that

$$0 \equiv u_{\bar{\sigma}(V)} \leq u_{\bar{\sigma}(V-1)} \leq \cdots \leq u_{\bar{\sigma}(k)} \leq \cdots u_{\bar{\sigma}(1)} \leq 1 \equiv u_{\bar{\sigma}(0)} \quad (11)$$

We have then the decomposition

$$x_{ij}^T(\{w\}) = \sum_{k=1}^V [u_{\bar{\sigma}(k-1)} - u_{\bar{\sigma}(k)}] B_k(T, \bar{\sigma})_{ij}. \quad (12)$$

which proves that  $x_{ij}^T(\{w\})$ , as a barycenter of positive type matrices with positives weights, is positive type.  $\square$

## 2.4 Example

Let us consider the graph  $G$  of Fig. 1. It has 6 edges  $\{l_1, l_2, l_3, l_4, l_5, l_6\}$  and 12 spanning trees:

$$\begin{aligned} &\{l_1, l_2, l_3\}, \{l_1, l_2, l_4\}, \{l_1, l_3, l_4\}, \{l_2, l_3, l_4\}, \{l_1, l_2, l_5\}, \{l_1, l_2, l_6\}, \\ &\{l_3, l_4, l_5\}, \{l_3, l_4, l_6\}, \{l_1, l_5, l_4\}, \{l_1, l_6, l_4\}, \{l_3, l_5, l_2\}, \{l_3, l_6, l_2\}. \end{aligned} \quad (13)$$

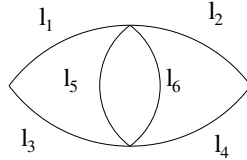


Figure 1: The Graph  $G$  with 6 edges and 12 spanning trees.

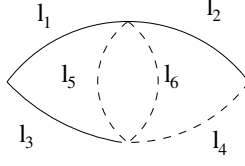


Figure 2: The spanning tree  $T_{123}$ . The dotted lines correspond to the loop lines.

Let us compute the constructive weights  $w(G, T)$  for each of these trees. To each edge  $l_i$  we associate a factor  $w_i$ . Consider first the spanning tree  $T_{123} = \{l_1, l_2, l_3\}$ , see Figure(2). The corresponding loop lines are  $l_4$ ,  $l_5$  and  $l_6$ . The weakening factor for  $l_5$  and  $l_6$  is  $\inf(w_1, w_3)$  and the weakening factor for  $l_4$  is  $\inf(w_1, w_2, w_3)$ . Therefore we have

$$\begin{aligned}
w(G, T_{123}) &= \int_{0 < w_1 < w_2 < w_3 < 1} dw_1 dw_2 dw_3 \inf(w_1, w_3)^2 \inf(w_1, w_2, w_3) \\
&+ \text{other permutations of } w_1, w_2, w_3 \\
&= \int_{w_1 < w_2 < w_3} dw_1 dw_2 dw_3 w_1^3 + \int_{w_2 < w_3 < w_1} dw_1 dw_2 dw_3 w_3^2 w_2 \\
&+ \int_{w_3 < w_1 < w_2} dw_1 dw_2 dw_3 w_3^3 + \int_{w_2 < w_1 < w_3} dw_1 dw_2 dw_3 w_1^2 w_2 \\
&+ \int_{w_3 < w_2 < w_1} dw_1 dw_2 dw_3 w_3^3 + \int_{w_1 < w_3 < w_2} dw_1 dw_2 dw_3 w_1^3.
\end{aligned}$$

We compute only two of the integrals explicitly as others are obtained by changing the names of variables.

$$\int_{w_1 < w_2 < w_3} dw_1 dw_2 dw_3 w_1^3 = \int_0^1 dw_3 \int_0^{w_3} dw_2 \int_0^{w_2} dw_1 w_1^3 = \frac{1}{120}, \quad (14)$$

$$\int_{w_1 < w_2 < w_3} dw_1 dw_2 dw_3 w_3^2 w_2 = \frac{1}{60}. \quad (15)$$

So we have

$$w(G', T_{123}) = \frac{1}{120} \times 4 + \frac{1}{60} \times 2 = \frac{1}{15}. \quad (16)$$

The constructive weights in  $G$  of the spanning trees  $T_{124}$ ,  $T_{134}$  and  $T_{234}$  are the same.

Next we consider the tree  $\{l_1, l_2, l_5\}$ . (See Figure 3). The weakening



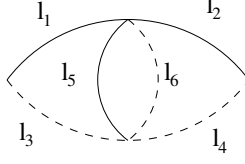


Figure 3: The spanning tree  $\{l_1, l_2, l_5\}$

factors are  $\inf(w_1, w_5)$  for loop line  $l_3$ ,  $\inf(w_2, w_5)$  for loop line  $l_4$  and  $w_5$  for loop line  $l_6$ . Hence

$$\begin{aligned}
w(G, T_{125}) &= \int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 \inf(w_1, w_5) \inf(w_2, w_5) w_5 \quad (17) \\
&+ \text{other permutations of } w_1, w_2, w_5 \\
&= \int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 + \int_{w_5 < w_1 < w_2} dw_1 dw_2 dw_5 w_5^3 \\
&+ \int_{w_2 < w_5 < w_1} dw_1 dw_2 dw_5 w_5^2 w_2 + \int_{w_2 < w_1 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 \\
&+ \int_{w_1 < w_5 < w_2} dw_1 dw_2 dw_5 w_5^2 w_1 + \int_{w_5 < w_2 < w_1} dw_1 dw_2 dw_5 w_5^3.
\end{aligned}$$

We have

$$\int_{w_1 < w_2 < w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 = \frac{1}{48}, \quad (18)$$

$$\int_{w_5 < w_1 < w_2} dw_1 dw_2 dw_5 w_5^3 = \frac{1}{120}, \quad (19)$$

$$\int_{w_2 < w_5 < w_1} dw_1 dw_2 dw_5 w_2 w_5^2 = \frac{1}{60}. \quad (20)$$

Similarly we get

$$w(G, T_{125}) = \frac{1}{120} \times 2 + \frac{1}{60} \times 2 + \frac{1}{48} \times 2 = \frac{11}{120}. \quad (21)$$

This is also the constructive weight of trees  $T_{126}, T_{345}, T_{346}, T_{125}, T_{145}, T_{146}, T_{235}$  and  $T_{236}$ .

We can check that

$$\sum_{T \in G} w(G, T) = 4 \cdot \frac{1}{15} + 8 \cdot \frac{11}{120} = 1. \quad (22)$$

We remark that  $6! = 720$ , hence that  $N(G', T_{123}) = 48$  and  $N(G', T_{125}) = 66$ . This can be checked by direct counting of the sectors  $\sigma$  with  $T(\sigma) = T_{123}$  or  $T(\sigma) = T_{125}$ . The 48 sectors with  $T(\sigma) = T_{123}$  are the thirty-six sectors with 123 as first three edges, plus the six sectors 135624, 136524, 135264, 135246, 136254, 136245 and the six analogs with 1 and 3 exchanged. The 66 sectors with  $T(\sigma) = T_{125}$  are the 36 with 123 as first three edges plus 30 others: six starting with 15 with third sector either 3 or 6; six analogs starting with 25 with third sector either 4 or 6; 6 starting with 51 with third sector either 3 or 6, 6 analogs starting with 15 with third sector either 3 or 6, and finally six sectors starting with 56 with third sector either 1 or 2.

## 3 The Graphs

### 3.1 Straightforward Repacking

Consider the expansion (1) of a connected quantity  $S$ . Reordering ordinary Feynman perturbation theory according to trees with relation (4) rearranges the Feynman expansion according to trees with the same number of vertices as the initial graph. Hence it reshuffles the various terms of a *given, fixed* order of perturbation theory. Remark that if the initial graphs have say degree 4 at each vertex, only trees with degree less than or equal to 4 occur in the rearranged tree expansion.

For Fermionic theories this is typically sufficient and one has for small enough coupling

$$\sum_T |A_T| < \infty \quad (23)$$

because Fermionic graphs mostly compensate each other at a fixed order by Pauli's principle; mathematically this is because these graphs form a determinant and the size of a determinant is much less than what its permutation expansion suggests. This is well known [25, 26, 27].

But this straightforward repacking cannot succeed for Bosonic theories, because we know the graphs at given order add up with the same sign! In this Bosonic case interesting reshufflings can only occur between graphs of different orders. This is precisely what the LVE does.

## 3.2 The Loop Vertex Expansion Repacking

The initial formulation of the loop vertex expansion [7] consists in applying the forest formula of [22, 23] to the intermediate field representation. As we explain now, it can also be reformulated as (4) but for the graphs of this intermediate field representation, which resum an infinite number of pieces of the ordinary graphs.

The principle of the intermediate field representation is to decompose any interaction of degree higher than three in terms of simpler three-body interactions. It is an extremely useful idea, with deep applications both to mathematics and physics. Quantum field theory, in particular, often discovered an intermediate field and its corresponding physical particles inside what was initially considered as local four body interactions<sup>5</sup>.

It is easy to describe the intermediate field method in terms of functional integrals, as it is a simple generalization of the formula

$$e^{-\lambda\phi^4/2} = \int e^{-\sigma^2/2} e^{i\sqrt{\lambda}\sigma\phi^2} d\sigma. \quad (24)$$

In this section we introduce the graphical procedure equivalent to this formula for the simple case of the  $\phi^4$  interaction.

In that case each vertex has exactly four half-lines. There are exactly three ways to pair these half-lines into two pairs. Hence each fully labeled (vacuum) graph of order  $n$  (with labels on vertices and half-lines), which has  $2n$  lines can be decomposed exactly into  $3^n$  labeled graphs  $G'$  with degree 3 and two different types of lines

- the  $2n$  old ordinary lines
- $n$  new dotted lines which indicate the pairing chosen at each vertex (see Figure 5).

Such graphs  $G'$  are called the 3-body extensions of  $G$  and we write  $G' \text{ ext } G$  when  $G'$  is an extension of  $G$ . Let us introduce for each such extension  $G'$  an amplitude  $A_{G'} = 3^{-n} A_G$  so that

$$A_G = \sum_{G' \text{ ext } G} A_{G'} \quad (25)$$

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<sup>5</sup>Recall that intermediate field representations are particularly natural for 4-body interactions but can be generalized to higher interactions as well [8].

when  $G'$  is an extension of  $G$ .

Now the ordinary lines of any extension  $G'$  of any  $G$  must form cycles. These cycles are joined by dotted lines.

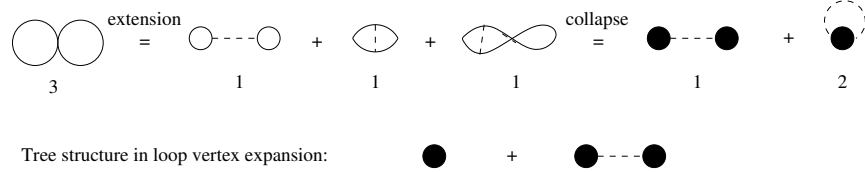


Figure 4: The extension and collapse for order 1 graph, with combinatoric weights shown below.

**Definition 3.1.** We define the collapse  $\bar{G}'$  of such a graph  $G'$  as the graph obtained by contracting each cycle to a "bold" vertex (see Figure 4). We write  $\bar{G}' \text{ coll } G'$  if  $\bar{G}'$  is the collapse of  $G'$ , and define the amplitude of the collapsed graph  $\bar{G}'$  as equal to that of  $G'$ . And  $\bar{T}$  is defined as the spanning tree of the collapsed graph  $\bar{G}'$ .

Remark that collapsed graphs, made of bold vertices and dotted lines, can have now arbitrary degree at each vertex. Remark also that several different extensions of a graph  $G$  can have different collapsed graphs, see Figure 4.

The loop vertex expansion rewrites

$$S = \sum_G A_G = \sum_{G' \text{ ext } G} A_{G'} = \sum_{\bar{G}' \text{ coll } G' \text{ ext } G} A_{\bar{G}'}. \quad (26)$$

Now we perform the tree repacking according to the graphs  $\bar{G}'$  with the  $n$  dotted lines and *not* with respect to  $G$ . This is a completely different repacking:

$$A_{\bar{G}'} = \sum_{\bar{T} \in \bar{G}'} w(\bar{G}', \bar{T}) A_{\bar{G}'}, \quad (27)$$

so that

$$S = \sum_{G' \text{ ext } G} A_{G'} = \sum_{\bar{T} \in \bar{G}'} A_{\bar{T}}, \quad (28)$$

$$A_{\bar{T}} = \sum_{\bar{G}' \supset \bar{T}} w(\bar{G}', \bar{T}) A_{\bar{G}'}. \quad (29)$$

**Theorem 3.1.** *For  $\lambda$  small*

$$\sum_{\bar{T}} |A_{\bar{T}}| < \infty \quad (30)$$

*the result being the Borel sum of the initial perturbative series [24].*

The proof of the theorem will not be recalled here (see [7, 9, 24]) but it relies on the positivity property of the  $x_\ell^T(\{w\})$  symmetric matrix, and the representation of each  $A_{\bar{T}}$  amplitude as an integral over a corresponding normalized Gaussian measure of a product of resolvents bounded by 1. This convergence would not be true if we had chosen naive  $w(T, G)$  equally distributed weights.

## 4 Examples of extensions and collapses

In this section we give the extension and collapse of the Feynman graphs for  $Z$  and  $\log Z$  for the  $\phi_0^4$  model up to order 2. We also recover the combinatorics of those graphs through the ordinary functional integral formula for the loop vertex expansion formula of [24].

The extension and collapse at order 1 was shown in Figure 4. In this case the tree structure is easy. We find only the trivial "empty" tree with one vertex and no edge and the "almost trivial" tree with two vertices and a single edge. The weight for these trees is 1.

At second order we find one disconnected Feynman graph and two connected ones. Only the connected ones survive in the expansion of  $\log Z$ .

The corresponding graphs and tree structures are shown in Figure 5 and 6. Using the loop vertex expansion formula we begin to see that graphs coming from different orders of the expansion of  $\lambda$  can be associated to the same tree by the loop vertex expansion. Indeed we recover contributions for the trivial and almost trivial trees of the previous figure. But we find also a new contribution belonging to a tree with two edges.

From these examples we find that the structure of the loop vertex expansion is totally different from that of Feynman graph calculus. At each order of the loop vertex expansion it combines terms in different orders of  $\lambda$ .

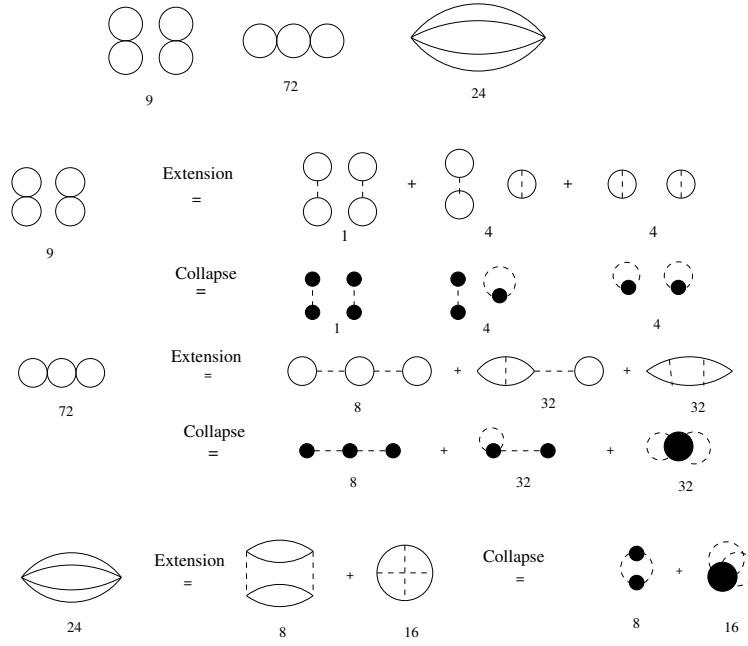


Figure 5: The extension and collapse for order 2 graph and their combinatorial factors.

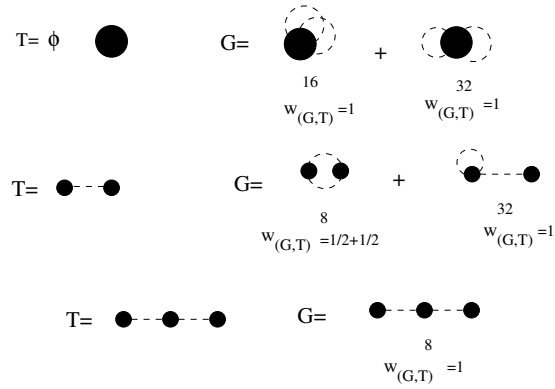


Figure 6: The connected graphs and the tree structure from the loop vertex expansion.

## 5 Non-integer Dimension

Let us now consider, e.g. for  $0 < D \leq 2$  the Feynman amplitudes for the  $\phi_D^4$  theory. They are given by the following convergent parametric representation (see e.g. [28] for a recent reference)

$$A_{D,G} = \int_0^\infty d\alpha \frac{e^{-m^2 \sum_\ell \alpha_\ell}}{U_G^{D/2}} \quad (31)$$

where  $m$  is the mass and  $U_G$  is the Kirchoff-Symanzik polynomial for  $G$

$$U_G = \sum_{T \in G} \prod_{\ell \notin T} \alpha_\ell. \quad (32)$$

All the previous decompositions working at the level of graphs, they are independent of the space-time dimension. We can therefore repack the series of Feynman amplitudes in non-integer dimension to get the  $D$  dimensional tree amplitude:

$$A_{D,T'} = \sum_{G \supset T'} w(T', G) A_{D,G}. \quad (33)$$

We know that for  $D = 0$  and  $D = 1$  the loop vertex expansion is convergent. Therefore it is tempting to conjecture, for instance at least for  $D$  real and  $0 \leq D < 2$  (that is when no ultraviolet divergences require renormalization)

**Conjecture 5.1.** *For  $\lambda$  small*

$$\sum_{T'} |A_{D,T'}| < \infty, \quad (34)$$

*the result being the Borel sum of the initial perturbation series.*

If true this conjecture would allow rigorous interpolation between quantum field theories in various dimensions of space time. It could e.g. lead to a possible justification of the Wilson-Fisher  $4 - \epsilon$  expansion that allows good numerical approximate computations of critical indices in 3 dimensions.

An other approach to quantum field theory in non integer dimension, also based on the forest formula but more radical, is proposed in [29].

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## References

- [1] F. Dyson, Divergence of perturbation theory in quantum electrodynamics, *Phys Rev.* 85, 631 (1952).
- [2] A. Jaffe, Divergence of perturbation theory for bosons, *Comm. Math. Phys.* 1, 127 (1965).
- [3] C. de Calan and V. Rivasseau, The perturbation series for  $\phi_3^4$  field theory is divergent, *Comm. Math. Phys.* 83, 77 (1982).
- [4] V. Rivasseau, “From perturbative to constructive renormalization,” *Princeton, USA: Univ. Pr. (1991) 336 p. (Princeton series in physics)*
- [5] A. Lindstedt, *Abh. K. Akad. Wiss. St. Petersburg* 31, No. 4 (1882); H. Poincaré H. (1957) [1893], *Les Méthodes Nouvelles de la Mécanique Céleste*, II, New York: Dover Publ.
- [6] J. Glimm and A. M. Jaffe, “Quantum Physics. A Functional Integral Point Of View,” *New York, Usa: Springer ( 1987) 535p*
- [7] V. Rivasseau, “Constructive Matrix Theory,” *JHEP* **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].
- [8] V. Rivasseau and Z.T. Wang, Loop Vertex Expansion for  $\phi^{2k}$  Theory in Zero Dimension, arXiv:1003.1037, *J. Math. Phys.* **51** (2010) 092304
- [9] J. Magnen and V. Rivasseau, “Constructive  $\phi^4$  field theory without tears,” *Annales Henri Poincare* **9** (2008) 403 [arXiv:0706.2457 [math-ph]].
- [10] V. Rivasseau and Z. Wang, “Constructive Renormalization for  $\Phi_2^4$  Theory with Loop Vertex Expansion,” *J. Math. Phys.* **53**, 042302 (2012) [arXiv:1104.3443 [math-ph]].
- [11] Z.T. Wang, Construction of 2-dimensional Grosse-Wulkenhaar Model, [arXiv:1104.3750 [math-ph]].
- [12] R. Gurau, “The  $1/N$  Expansion of Tensor Models Beyond Perturbation Theory,” arXiv:1304.2666 [math-ph].



- [13] R. Gurau, “The  $1/N$  expansion of colored tensor models,” *Annales Henri Poincaré* **12**, 829 (2011) [arXiv:1011.2726 [gr-qc]].
- [14] R. Gurau and V. Rivasseau, “The  $1/N$  expansion of colored tensor models in arbitrary dimension,” *Europhys. Lett.* **95**, 50004 (2011) [arXiv:1101.4182 [gr-qc]].
- [15] R. Gurau, “The complete  $1/N$  expansion of colored tensor models in arbitrary dimension,” *Annales Henri Poincaré* **13**, 399 (2012) [arXiv:1102.5759 [gr-qc]].
- [16] R. Gurau and J. P. Ryan, “Colored Tensor Models - a review,” *SIGMA* **8**, 020 (2012) [arXiv:1109.4812 [hep-th]].
- [17] R. Gurau, “Universality for Random Tensors,” arXiv:1111.0519 [math.PR].
- [18] V. Bonzom, R. Gurau and V. Rivasseau, “Random tensor models in the large  $N$  limit: Uncoloring the colored tensor models,” *Phys. Rev. D* **85**, 084037 (2012) [arXiv:1202.3637 [hep-th]].
- [19] V. Rivasseau, “Quantum Gravity and Renormalization: The Tensor Track,” *AIP Conf. Proc.* **1444**, 18 (2011) [arXiv:1112.5104 [hep-th]].
- [20] V. Rivasseau, “The Tensor Track: an Update,” arXiv:1209.5284 [hep-th].
- [21] K. Hepp, *Théorie de la renormalisation*, Berlin, Springer Verlag, 1969.
- [22] D. Brydges and T. Kennedy, Mayer expansions and the Hamilton-Jacobi equation, *Journal of Statistical Physics*, **48**, 19 (1987).
- [23] A. Abdesselam and V. Rivasseau, “Trees, forests and jungles: A botanical garden for cluster expansions,” arXiv:hep-th/9409094.
- [24] V. Rivasseau, *Constructive Field Theory in Zero Dimension*, arXiv:0906.3524, *Advances in Mathematical Physics*, Volume 2009 (2009), Article ID 180159
- [25] A. Lesniewski, Effective Action for the Yukawa<sub>2</sub> Quantum Field Theory, *Commun. Math. Phys.* **108**, 437 (1987).

- [26] J. Feldman, J. Magnen, V. Rivasseau and E. Trubowitz, An Infinite Volume Expansion for Many Fermion Green's functions, *Helv. Phys. Acta*, **65**, 679 (1992).
- [27] A. Abdesselam and V. Rivasseau, Explicit Fermionic Cluster Expansion, *Lett. Math. Phys.* **44**, 77-88 (1998), [arXiv:cond-mat/9712055](#).
- [28] T. Krajewski, V. Rivasseau, A. Tanasa and Z.T. Wang, "Topological Graph Polynomials and Quantum Field Theory, Part I: Heat Kernel Theories," *Journal of Noncommutative Geometry.* **4**, 29-82 (2010) [arXiv:0811.0186 \[math-ph\]](#).
- [29] R. Gurau, J. Magnen and V. Rivasseau, "Tree Quantum Field Theory," *Annales Henri Poincare* **10** (2009) 867 [[arXiv:0807.4122 \[hep-th\]](#)].